Solution to Exercise 6[1]

1. (a) $\mathbf{x}_u = (2u, 2v, 2u)$, $\mathbf{x}_v = (-2v, 2u, 2v)$.

Hence, $\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2u & 2v & 2u \\ -2v & 2u & 2v \end{vmatrix} = (4(v^2 - u^2), -8uv, 4(u^2 + v^2))$.

$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{4(v^2 - u^2)^2 + (-8uv)^2 + [4(u^2 + v^2)]^2} = 4\sqrt{2}(u^2 + v^2)$.

Therefore, $\hat{\mathbf{n}} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \dfrac{(4(v^2 - u^2), -8uv, 4(u^2 + v^2))}{4\sqrt{2}(u^2 + v^2)} = \dfrac{1}{\sqrt{2}}(\dfrac{v^2 - u^2}{u^2 + v^2}, -\dfrac{2uv}{u^2 + v^2}, 1)$.

Note that differentiating $\mathbf{x}_u$ and $\mathbf{x}_v$ are much easier, we may find $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$.

$\mathbf{x}_{uu} = (2, 0, 2)$, $\mathbf{x}_{uv} = (0, 2, 0)$, $\mathbf{x}_{vv} = (-2, 0, 2)$.

Next, $e = \langle \mathbf{x}_{uu}, \hat{\mathbf{n}} \rangle = \dfrac{2\sqrt{2}v^2}{u^2 + v^2}$, $f = \langle \mathbf{x}_{uv}, \hat{\mathbf{n}} \rangle = -\dfrac{2\sqrt{2}uv}{u^2 + v^2}$, $g = \langle \mathbf{x}_{vv}, \hat{\mathbf{n}} \rangle = \dfrac{2\sqrt{2}u^2}{u^2 + v^2}$.

Therefore $\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \dfrac{2\sqrt{2}}{u^2 + v^2} \begin{pmatrix} v^2 & -uv \\ -uv & u^2 \end{pmatrix}$.

(b) $\mathbf{x}_u = (1, 0, v)$, $\mathbf{x}_v = (0, 1, u)$.

Hence, $\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$.

$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{(-v)^2 + (-u)^2 + 1^2} = \sqrt{u^2 + v^2 + 1}$.

Therefore, $\hat{\mathbf{n}} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \dfrac{(-v, -u, 1)}{\sqrt{u^2 + v^2 + 1}}$.

Note that differentiating $\mathbf{x}_u$ and $\mathbf{x}_v$ are much easier, we may find $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$.

$\mathbf{x}_{uu} = (0, 0, 0)$, $\mathbf{x}_{uv} = (0, 0, 1)$, $\mathbf{x}_{vv} = (0, 0, 0)$.

Next, $e = \langle \mathbf{x}_{uu}, \hat{\mathbf{n}} \rangle = 0$, $f = \langle \mathbf{x}_{uv}, \hat{\mathbf{n}} \rangle = -\dfrac{u}{\sqrt{u^2 + v^2 + 1}}$, $g = \langle \mathbf{x}_{vv}, \hat{\mathbf{n}} \rangle = 0$.

Therefore $\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = -\dfrac{u}{\sqrt{u^2 + v^2 + 1}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(c) $\mathbf{x}_u = (3u^2 \cos v, 3u^2 \sin v, 1)$, $\mathbf{x}_v = (-u^3 \sin v, u^3 \cos v, 0)$.

Hence, $\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3u^2 \cos v & 3u^2 \sin v & 1 \\ -u^3 \sin v & u^3 \cos v & 0 \end{vmatrix} = (-u^3 \cos v, -u^3 \sin v, 3u^5)$.

$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{(-u^3 \cos v)^2 + (-u^3 \sin v)^2 + (3u^5)^2} = u^3\sqrt{9u^4 + 1}$ as $u > 0$.

Therefore, $\hat{\mathbf{n}} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \dfrac{(-\cos v, -\sin v, 3u^2)}{\sqrt{9u^4 + 1}}$.

Since differentiating $\mathbf{x}_u$ and $\mathbf{x}_v$ are relatively easier, we may find $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$.

$\mathbf{x}_{uu} = (6u \cos v, 6u \sin v, 0)$, $\mathbf{x}_{uv} = (-3u^2 \sin v, 3u^2 \cos v, 0)$, $\mathbf{x}_{vv} = (-u^3 \cos v, -u^3 \sin v, 0)$.

Next, $e = \langle \mathbf{x}_{uu}, \hat{\mathbf{n}} \rangle = -\dfrac{6u}{\sqrt{9u^4 + 1}}$, $f = \langle \mathbf{x}_{uv}, \hat{\mathbf{n}} \rangle = 0$, $g = \langle \mathbf{x}_{vv}, \hat{\mathbf{n}} \rangle = \dfrac{u^3}{\sqrt{9u^4 + 1}}$.

Therefore $\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \dfrac{u}{\sqrt{9u^4 + 1}} \begin{pmatrix} -6 & 0 \\ 0 & u^2 \end{pmatrix}$.

[1] If you have any questions or spot any mistakes, find feel free to contact Michael Cheung through **michaelcheung0723@gmail.com** for Questions 1 to 3, and Max Shung through **maxshung.math@gmail.com** for Questions 4 to 5.

2. (a) Since $F = 0$, by considering Theorem 3.3.11 in lecture notes, we have

$$K = -\frac{1}{2\sqrt{EG}}[(\frac{E_v}{\sqrt{EG}})_v + (\frac{G_u}{\sqrt{EG}})_u] = -\frac{1}{2f^2}[(\frac{2ff_v}{f^2})_v + (\frac{2ff_u}{f^2})_u]$$
$$= -\frac{1}{f^2}[(\frac{f_u}{f})_u + (\frac{f_v}{f})_v] = -\frac{1}{f^2}[(\ln f)_{uu} + (\ln f)_{vv}] = -\frac{1}{f^2}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})\ln f$$

(b) Note that $\frac{1}{\sqrt{u^2+v^2+1}} > 0$.

$$K = -(u^2+v^2+1)(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})\ln(\frac{1}{\sqrt{u^2+v^2+1}})$$
$$= \frac{u^2+v^2+1}{2}[\frac{-2u^2+2v^2+2}{(u^2+v^2+1)^2} + \frac{2u^2-2v^2+2}{(u^2+v^2+1)^2}] = \frac{2}{u^2+v^2+1}$$

(c) Note that $e^{-u} > 0$.

$$K = -e^{u^2}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})\ln(e^{-\frac{u^2}{2}}) = -e^{u^2}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})(-\frac{u^2}{2}) = e^{u^2}$$

3. (a) $\mathbf{x}_u = (\cos v, \sin v, 0)$, $\mathbf{x}_v = (-u\sin v, u\cos v, 1)$.
Hence, $\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v, -\cos v, u)$.
$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{(\sin v)^2 + (-\cos v)^2 + u^2} = \sqrt{u^2+1}$.
Therefore, $\hat{\mathbf{n}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{(\sin v, -\cos v, u)}{\sqrt{u^2+1}}$.

(b) Since $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$, $\mathbf{x}(u,v)$ is an orthogonal parametrization.

(c) Note that $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1$, $F = 0$, $G = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = u^2 + 1$.
Also, $\mathbf{x}_{uu} = (0,0,0)$, $\mathbf{x}_{uv} = (-\sin v, \cos v, 0)$, $\mathbf{x}_{vv} = (-u\cos v, -u\sin v, 0)$.
Hence, $e = \langle \mathbf{x}_{uu}, \hat{\mathbf{n}} \rangle = 0$, $f = \langle \mathbf{x}_{uv}, \hat{\mathbf{n}} \rangle = -\frac{1}{\sqrt{u^2+1}}$, $g = \langle \mathbf{x}_{vv}, \hat{\mathbf{n}} \rangle = 0$. As a result, $K = \frac{\det\ \mathbf{II}}{\det\ \mathbf{I}} = \frac{eg - f^2}{EG - F^2} = \frac{-\frac{1}{u^2+1}}{u^2+1} = -\frac{1}{(u^2+1)^2}$.

4. (a) S is a minimal surface if the mean curvature $H = 0$ at every points of S.

   (b) We see that the mean curvatures of Helicoid and Catenoid are both zero, at everywhere. Thus, Helicoid and Catenoid are minimal surfaces.

   (c) i. Parametrize the surface Φ by

$$X(\theta, z) = (f(z)\cos\theta, f(z)\sin\theta, z), \text{ for } (\theta, z) \in (0, 2\pi) \times \mathbb{R}$$

Then, we have

$$\begin{cases} X_\theta = (-f(z)\sin\theta, f(z)\cos\theta, 0) \\ X_z = (f'(z)\cos\theta, f'(z)\sin\theta, 1) \end{cases}$$

The first fundamental form of the surface is given by

$$\mathbf{I} = \begin{pmatrix} (f(z))^2 & 0 \\ 0 & (f'(z))^2 + 1 \end{pmatrix}$$

Consider the unit normal vector

$$\begin{aligned} \mathbf{n} = \frac{X_\theta \times X_z}{\|X_\theta \times X_z\|} &= \frac{1}{f(z)\sqrt{1 + (f'(z))^2}} (f(z)\cos\theta, f(z)\sin\theta, -f(z)f'(z)) \\ &= \pm\frac{1}{\sqrt{1 + (f'(z))^2}} (\cos\theta, \sin\theta, -f'(z)) \end{aligned}$$

Also, we have

$$\begin{cases} X_{\theta\theta} = (-f(z)\cos\theta, -f(z)\sin\theta, 0) \\ X_{\theta z} = (-f'(z)\sin\theta, f'(z)\cos\theta, 0) = X_{z\theta} \\ X_{zz} = (f''(z)\cos\theta, f''(z)\sin\theta, 0) \end{cases}$$

With respect to the normal vector $\mathbf{n}$, the second fundamental form is given by

$$\mathbf{II} = \pm\frac{1}{\sqrt{1 + (f'(z))^2}} \begin{pmatrix} -f(z) & 0 \\ 0 & f''(z) \end{pmatrix}$$

Therefore, the Gaussian curvature of the surface Φ is given by

$$K_\Phi(z) = \frac{\det(\mathbf{II})}{\det(\mathbf{I})} = \frac{-f(z)f''(z)/\left(1 + (f'(z))^2\right)}{(f(z))^2\left(1 + (f'(z))^2\right)} = -\frac{f''(z)}{f(z)\left(1 + (f'(z))^2\right)^2}$$

Also, the mean curvature of the surface Φ is

$$\begin{aligned} H_\Phi(z) &= \frac{1}{2} \cdot \frac{1}{\sqrt{1 + (f'(z))^2}} \left(\frac{f''(z)(f(z))^2 - 0 + \left(1 + (f'(z))^2\right)(-f(z))}{(f(z))^2\left(1 + (f'(z))^2\right)} \right) \\ &= \pm\frac{1}{2(f(z))^2\left(1 + (f'(z))^2\right)^{\frac{3}{2}}} \left(f(z)\left[f(z)f''(z) - 1 - (f'(z))^2\right] \right) \\ &= \pm\frac{1 + (f'(z))^2 - f(z)f''(z)}{2f(z)\left(1 + (f'(z))^2\right)^{\frac{3}{2}}} \end{aligned}$$

Remark. The original problem set omitted the " $\pm$ " sign. The mean curvature depends on which unit normal vector you pick.

ii. Define

$$f(z) = a\left(\cosh\frac{z}{a}\cosh b + \sinh\frac{z}{a}\sinh b\right) = a\cosh\left(\frac{z}{a} + b\right)$$

Note that

$$f'(z) = \sinh\left(\frac{z}{a} + b\right) \text{ and } f''(z) = \frac{1}{a}\cosh\left(\frac{z}{a} + b\right)$$

Then, we have

$$\begin{aligned} 1 + (f'(z))^2 - f(z)f''(z) &= 1 + \sinh^2\left(\frac{z}{a} + b\right) - a\cdot\frac{1}{a}\cosh^2\left(\frac{z}{a} + b\right) \\ &= \cosh^2\left(\frac{z}{a} + b\right) - \cosh^2\left(\frac{z}{a} + b\right) \\ &= 0 \end{aligned}$$

with $a > 0$ and $b \in \mathbb{R}$. Also, we have $f(z) > 0$ and $1 + (f'(z))^2 > 0$, hence by using part (c)i., the mean curvature of the surface obtained by rotating the graph of $x = f(z)$ is

$$H = \frac{1 + (f'(z))^2 - f(z)f''(z)}{2f(z)\left(1 + (f'(z))^2\right)^{\frac{3}{2}}} \equiv 0$$

everywhere. Thus, using part (a), the surface is a minimal surface.

5. (a) Note that

$$\begin{cases} \langle \mathbf{x}_u, \mathbf{n}_u \rangle = a_{11} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + a_{12} \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_u, \mathbf{n}_v \rangle = a_{21} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + a_{22} \langle \mathbf{x}_u, \mathbf{x}_v \rangle \end{cases} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\ \langle \mathbf{x}_u, \mathbf{x}_v \rangle \end{pmatrix}$$

Also, we have

$$\begin{cases} \langle \mathbf{x}_v, \mathbf{n}_u \rangle = a_{11} \langle \mathbf{x}_v, \mathbf{x}_u \rangle + a_{12} \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{n}_v \rangle = a_{21} \langle \mathbf{x}_v, \mathbf{x}_u \rangle + a_{22} \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{cases} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_v, \mathbf{x}_u \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}$$

Therefore, it follows that

$$\begin{pmatrix} \langle \mathbf{x}_u, \mathbf{n}_u \rangle & \langle \mathbf{x}_u, \mathbf{n}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{n}_u \rangle & \langle \mathbf{x}_v, \mathbf{n}_v \rangle \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}$$

$$-(\mathbf{II}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{I}$$

Therefore, we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -(\mathbf{II})(\mathbf{I})^{-1}$$

(b) The differential of the Gauss map is defined by

$$\begin{cases} d\mathbf{n}_p (\mathbf{x}_u) = \mathbf{n}_u \\ d\mathbf{n}_p (\mathbf{x}_v) = \mathbf{n}_v \end{cases}$$

and its matrix representation is given by

$$d\mathbf{n}_p = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

As the shape operator of $\mathbf{X}$ is the negative differential of the Gauss map, the matrix representation is defined as follows:

$$-d\mathbf{n}_p = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Thus, from part (a), we have

$$S = -d\mathbf{n}_p = (\mathbf{II})(\mathbf{I})^{-1}$$

(c) Note that

$$K = \frac{\det(\mathbf{II})}{\det(\mathbf{I})} = \det\left((\mathbf{II})(\mathbf{I})^{-1}\right) = \det(S)$$

and

$$
\begin{aligned}
H &= \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) \\
&= \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\right] \\
&= \frac{1}{2}\,\mathrm{tr}\left((\mathbf{II})(\mathbf{I})^{-1}\right) = \frac{1}{2}\,\mathrm{tr}(S)
\end{aligned}
$$

(d) By direct computation, we have

$$
\begin{aligned}
A^2 - \mathrm{tr}(A)A + \det(A)I &= \begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 - (a+c)\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \left(ac - b^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{pmatrix} - \begin{pmatrix} a^2 + ac & ab + bc \\ ab + bc & ac + c^2 \end{pmatrix} + \begin{pmatrix} ac - b^2 & 0 \\ 0 & ac - b^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}
\end{aligned}
$$

Remark. See Cayley Hamilton's Theorem.

Hence, from (*) and part (c), we have

$$
\begin{aligned}
S^2 - 2HS + KI &= S^2 - 2\left(\frac{1}{2}\,\mathrm{tr}(S)\right)S + \det(S)I \qquad \text{(from part (c))} \\
&= S^2 - \mathrm{tr}(S)S + \det(S)I \\
&= \mathbf{0} \qquad \text{(from(*))}
\end{aligned}
$$

(e) i. $K(\mathbf{p}) = \det(S) = (-2)^2 - 1 = 3$,

$$
H(\mathbf{p}) = \frac{1}{2}\,\mathrm{tr}(S) = \frac{1}{2}(-2-2) = -2
$$

ii. Consider

$$
\begin{aligned}
\det(S - \kappa I) &= 0 \\
\kappa^2 - 2(-2)\kappa + 3 &= 0 \\
(\kappa + 1)(\kappa + 3) &= 0 \\
\kappa = -1 \quad &\text{or } \kappa = -3
\end{aligned}
$$

Thus, the principal curvatures of $\mathbf{X}$ at $\mathbf{p}$ are $\kappa_1 = -3$ and $\kappa_2 = -1$.

iii. When $\kappa = \kappa_1 = -3$, then we have

$$
\begin{aligned}
(S + 3I)\boldsymbol{\xi}_1 &= \mathbf{0} \\
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\boldsymbol{\xi}_1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

By solving the simultaneous equations, we have

$$\boldsymbol{\xi}_1 = \left\{ s\begin{pmatrix} 1 \\ -1 \end{pmatrix} : s \in \mathbb{R}\backslash\{0\} \right\}$$

Thus, $\boldsymbol{\xi}_1 = (1, -1)$ is the principal direction associated with the principal curvature is $\kappa_1 = -3$.

- When $\kappa = \kappa_2 = -1$, then we have

$$(S + I)\boldsymbol{\xi}_2 = \mathbf{0}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By solving the system again, it follows that

$$\boldsymbol{\xi}_2 \in \left\{ t\begin{pmatrix} 1 \\ 1 \end{pmatrix} : t \in \mathbb{R}\backslash\{0\} \right\}$$

This implies that $\xi_2 = (1, 1)$ is the principal direction associated with the principal curvature is $\kappa_1 = -1$.

iv. From part (e) iii., we have

$$\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \rangle = 1(-1) + 1(1) = 0$$

Hence, two principal directions are orthogonal.
For any unit vector $\mathbf{v} \in T_{\mathbf{p}}(\mathbf{X}(u, v))$, the normal curvature of the surface at $\mathbf{p}$ along $\mathbf{v}$ is defined by

$$\kappa_n(\mathbf{v}) = -\langle \mathbf{v}, d\mathbf{n}_{\mathbf{p}}(\mathbf{v}) \rangle$$

As $T_{\mathbf{p}}(\mathbf{X}) = \operatorname{span}\{\mathbf{X}_u, \mathbf{X}_v\} = \operatorname{span}\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$ and $\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \rangle = 0$, we write

$$\mathbf{v} = \frac{1}{\sqrt{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}} (a\boldsymbol{\xi}_1 + b\boldsymbol{\xi}_2)$$

since $\mathbf{v}$ is a unit vector.
Then, we have

$$\begin{aligned}
\kappa_n(\mathbf{v}) &= -\frac{1}{\left(\sqrt{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\right)^2} \langle a\boldsymbol{\xi}_1 + b\boldsymbol{\xi}_2, d\mathbf{n}_{\mathbf{p}}(a\boldsymbol{\xi}_1 + b\boldsymbol{\xi}_2) \rangle \\
&= \frac{1}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2} \langle a\boldsymbol{\xi}_1 + b\boldsymbol{\xi}_2, a\kappa_1\boldsymbol{\xi}_1 + b\kappa_2\boldsymbol{\xi}_2 \rangle \\
&= \frac{a^2 \|\boldsymbol{\xi}_1\|^2}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\kappa_1 + \frac{b^2 \|\boldsymbol{\xi}_2\|^2}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\kappa_2
\end{aligned}$$

Also, from our assumption, we have $\kappa_1 \leq \kappa_2$, thus it follows that

$$\kappa_n(\mathbf{v}) \leq \frac{a^2 \|\boldsymbol{\xi}_1\|^2}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\kappa_2 + \frac{b^2 \|\boldsymbol{\xi}_2\|^2}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\kappa_2 = \kappa_2$$

and

$$\kappa_n(\mathbf{v}) \geq \frac{a^2 \|\boldsymbol{\xi}_1\|^2}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\kappa_1 + \frac{b^2 \|\boldsymbol{\xi}_2\|^2}{a^2 \|\boldsymbol{\xi}_1\|^2 + b^2 \|\boldsymbol{\xi}_2\|^2}\kappa_1 = \kappa_1$$

Thus, we have

$$-3 = \kappa_1 \leq \kappa_n(\mathbf{v}) \leq \kappa_2 = -1.$$